Complex and Imaginary Numbers

A. 1. $i^{57} = i(remainder.25)$ 2. $i^{122} = -1(remainder.5)$ 3. $i^{451} = -i(remainder.75)$

4. $5i + 6 - 3i - 5 = 2i + 1$

5. $2xi + 3i - 7xi + 5i = -5xi + 8i$

6. $i^5 + i^{19} - i^{28} = i - i - 1 = -1$

7. $5i(7 - 4i) = 35i - 20i^2 = 35i + 20$

8. $(3i^5)(2i^9)(i^{13}) = (3i)(2i)i = 6i^3 = -6i$

9. $(3 - 5i)^2 = (3 - 5i)(3 - 5i) = 9 - 15i - 15i + 25i^2 = 9 - 15i - 15i - 25 = -16 - 30i$

10. $(4i + 3)(2i - 7) = 8i^2 - 28i + 6i - 21 = -8 - 28i + 6i - 21 = -29 - 22i$

11. $\frac{2}{i} = \frac{2}{i} \cdot \frac{i}{i} = \frac{2i}{i^2} = -2i$

12. $\frac{1}{i^3} - \frac{3}{i^5} + \frac{1}{i^{17}} = \frac{1}{-i} - \frac{3}{i} + \frac{1}{i} = -\frac{1}{i} - \frac{3}{i} + \frac{1}{i} = -\frac{3}{i} \cdot \frac{i}{i} = -\frac{3i}{i^2} = 3i$

13. $\frac{5i}{2i - 3} = \frac{5i}{(2i - 3)} \cdot \frac{(2i + 3)}{(2i + 3)} = \frac{10i^2 + 15i}{4i^2 - 9} = \frac{-10 + 15i}{-13}$

14. $\frac{i + 3}{3i + 2} = \frac{(i + 3)}{(3i + 2)} \cdot \frac{(3i - 2)}{(3i - 2)} = \frac{3i^2 - 2i + 9i - 6}{9i^2 - 4} = \frac{-9 + 7i}{-13}$

B. 1. $\sqrt{-3} \cdot \sqrt{-5} = i\sqrt{3} \cdot i\sqrt{5} = i^2\sqrt{15} = -\sqrt{15}$

2. $3i\sqrt{-4} \cdot 2i\sqrt{-5} = 3i^2\sqrt{4} \cdot 2i^2\sqrt{5} = -3 \cdot 2 \cdot -2 \cdot \sqrt{5} = 12\sqrt{5}$

3. $\frac{5}{\sqrt{-3}} = \frac{5}{i\sqrt{3}} \cdot \frac{i\sqrt{3}}{i\sqrt{3}} = \frac{5i\sqrt{3}}{i^2 \cdot 3} = \frac{5i\sqrt{3}}{-3} = -\frac{5i\sqrt{3}}{3}$

4. $\frac{\sqrt{-6}}{\sqrt{-2}} = \frac{i\sqrt{6}}{i\sqrt{2}} = \sqrt{3}$

5. $\sqrt{-3}(\sqrt{-7} + 2i) = i\sqrt{3}(i\sqrt{7} + 2i) = i^2\sqrt{21} + 2i^2\sqrt{3} = -\sqrt{21} - 2\sqrt{3}$

6. $\frac{\sqrt{5}}{3 - \sqrt{-2}} = \frac{\sqrt{5}}{(3 - i\sqrt{2})} \cdot \frac{(3 + i\sqrt{2})}{(3 + i\sqrt{2})} = \frac{3\sqrt{5} + i\sqrt{10}}{9 - i^2 \cdot 2} = \frac{3\sqrt{5} + i\sqrt{10}}{11}$

7. $2\sqrt{-8} + 5\sqrt{-50} = 2i\sqrt{8} + 5i\sqrt{50} = 4i\sqrt{2} + 25i\sqrt{2} = 29i\sqrt{2}$

8. $4i\sqrt{-2} - 3\sqrt{-27} + 8i^2\sqrt{2} - \sqrt{-48} = 4i^2\sqrt{2} - 3i\sqrt{27} + 8i^2\sqrt{2} - i\sqrt{48} =$
$-4\sqrt{2} - 9i\sqrt{3} - 8\sqrt{2} - 4i\sqrt{3} = -12\sqrt{2} - 13i\sqrt{3}$

C. 1.
$$4xi = 7$$
$$x = \frac{7}{4i}$$
$$x = \frac{7}{4i}\cdot\frac{i}{i} = \frac{7i}{4i^2} = \frac{7i}{-4} = -\frac{7i}{4}$$

2.
$$5xi + 7 = -4i$$
$$5xi = -4i - 7$$
$$x = \frac{-4i - 7}{5i}$$
$$x = \frac{(-4i-7)}{5i}\cdot\frac{i}{i} =$$
$$x = \frac{-4i^2 - 7i}{5i^2} =$$
$$x = \frac{4 - 7i}{-5}$$

3.
$$2x + 3 - 7 = -5xi + 6i + 4$$
$$2x + 5xi = 6i + 4 + 4$$
$$x(2 + 5i) = 6i + 8$$
$$x = \frac{(6i+8)}{(2+5i)}\cdot\frac{(2-5i)}{(2-5i)}$$
$$x = \frac{12i - 30i^2 + 16 - 40i}{4 - 25i^2}$$
$$x = \frac{-28i + 46}{29}$$

4.
$$2xi + 5y + 3 + 2i = 7xi + 3i - 4 - 4y$$
$$2xi + 2i = 7xi + 3i \Leftrightarrow 5y + 3 = -4 - 4y$$
$$-5xi = i \qquad 9y = -7$$
$$x = \frac{i}{-5i} = -\frac{1}{5} \qquad y = -\frac{7}{9}$$

5.
$$3x - 7yi + 2 = 5y + 6x + 2i$$
$3x + 2 = 5y + 6x \Leftrightarrow -7yi = 2i$ (solve this eq. first)
$$-3x - 5y = -2 \qquad y = \frac{2i}{-7i} = -\frac{2}{7}$$
$3x + 5y = 2$ (substitute for "y")
$$3x + 5\left(-2/7\right) = 2$$
$$21x - 10 = 24 \Rightarrow 21x = 24 \Rightarrow x = 24/21 = 8/7$$

6.
$$5x + 2yi - 4 = 2y + 3xi + 6$$
$$5x - 4 = 2y + 6 \Leftrightarrow 2yi = 3xi$$
$5x - 2y = 10 \Leftrightarrow 2y = 3x \Rightarrow -3x + 2y = 0$ (solve as a system)

$$3(5x - 2y = 10) \Rightarrow \quad 15x - 6y = 30$$
$$5(-3x + 2y = 0) \Rightarrow -15x + 10y = 0$$
$$-3x + 2(15/2) = 0 \qquad 4y = 30$$
$$-6x + 30 = 0 \qquad y = 30/4 = 15/2$$
$$-6x = -30$$
$$x = 5$$

D. Check notes in reference section

$3x^2 - 7x + 6 = 0$

Nature: $b^2 - 4ac = (-7)^2 - 4(3)(6) = 49 - 72 = -23$ (2 imaginary roots)

E. a) Sum: $-b/a = -(-7)/3 = 7/3$, Product: $c/a = 6/3 = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(6)}}{2(3)} = \frac{7 \pm \sqrt{-23}}{6} = \frac{7 \pm i\sqrt{23}}{6}$$

$-4x^2 + 3x + 5 = 0$

Nature: $b^2 - 4ac = (3)^2 - 4(-4)(5) = 9 + 80 = 89$ (2 real roots)

b) Sum: $-b/a = -(3)/(-4) = 3/4$, Product: $c/a = 5/(-4) = -5/4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{(3)^2 - 4(-4)(5)}}{2(-4)} = \frac{-3 \pm \sqrt{89}}{-8}$$

$2x^2 + x + 5 = 0$

Nature: $b^2 - 4ac = (1)^2 - 4(2)(5) = 1 - 40 = -39$ (2 imaginary roots)

c) Sum: $-b/a = -1/2$, Product: $c/a = 5/2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(2)(5)}}{2(2)} = \frac{-1 \pm \sqrt{-39}}{4} = \frac{-1 \pm i\sqrt{39}}{4}$$

F. a) $\{-6, 7\} \Rightarrow x = -6$ and $x = 7 \Rightarrow (x+6)(x-7) = 0 \Rightarrow x^2 - x - 42 = 0$

b) $\{-6i, 6i\} \Rightarrow x = -6i$ and $x = 6i \Rightarrow (x+6i)(x-6i) = 0 \Rightarrow$
$x^2 - 6ix + 6ix - 36i^2 = 0 \Rightarrow x^2 + 36 = 0$

c)
$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$
$$x^2 - [(4-7i) + (4+7i)]x + (4-7i)(4+7i) = 0$$
$$x^2 - 8x + 16 - 49i^2 = 0$$
$$x^2 - 8x + 65 = 0$$

d)
$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$
$$x^2 - \left[\frac{-3-2i}{5} + \frac{-3+2i}{5}\right]x + \left(\frac{-3-2i}{5}\right)\left(\frac{-3+2i}{5}\right) = 0$$
$$x^2 - \left(-6/5\right)x + \frac{-9-4i^2}{25} = 0 \Rightarrow x^2 + 6/5\,x + \frac{13}{25} = 0$$
$$25x^2 + 30x + 13 = 0$$